Theory of 'self-similarity' of periodic approximants to a quasilattice

1991 J. Phys. A: Math. Gen. 24 4873

(http://iopscience.iop.org/0305-4470/24/20/019)

J. Phys. A: Math. Gen. 24 (1991) 4873-4887. Printed in the UK

Theory of 'self-similarity' of periodic approximants to a quasilattice

Komajiro Niizeki
Department of Physics, Tohoku University, Sendai 980, Japan

Received 3 June 1991

Abstract. It is shown that self-similarity of a quasilattice has a profound effect on the periodic approximants to the quasilattice: The periodic approximants are grouped into series so that (i) each series is generated from its prototype by a successive application of the deflation-and-rescaling, (ii) the space group is common among the members of the series and (iii) the unit cell of the approximant is scaled up by τ with the series number, where τ is the scale of self-similarity of the relevant quasilattice. These results are exemplified with application to the case of the octagonal quasilattice.

1. Introduction

The structure of a quasicrystal is described with a quasilattice (QL), while that of its approximant crystal is described with a periodic approximant (PA) to the QL (Elser and Henley 1985). We have developed a theory of the space groups of the PAs to a QL (Niizeki 1991a, b); a QL has PAs with different lattice constants and different space groups.

Self-similarity is one of the remarkable properties of a QL (see, for example, Niizeki 1989a). That is, if we select lattice points whose environments agree with one of a set of specified environments, the resulting set of points form another QL which is locally isomorphic to the original one except for the scale. The scale τ of self-similarity is equal to a PV-unit of the algebraic field relevant to the QL. In contrast, a PA to a QL cannot have self-similarity because it is periodic. There exist, however, a group of PAs whose unit cells are similar and the lattice constants of different members are scaled by powers of τ (Elser and Henley 1985, Duneau, Mosseri and Oguey 1989, herafter referred to as DMO). In this paper, we will show that there exists a more precise relation among the members of the group.

A QL is obtained by the cut-and-projection method from a mother lattice which is a periodic lattice of higher dimensionality than the physical dimension (see, for example, Janssen 1988); the mother lattice is cut with a strip before being projected onto the physical space. Similarly, a PA to the QL is obtained by the same method from its mother lattice, which is obtained by introducing a phason strain into the mother lattice of the QL (Ishii 1989). The phason strain makes a lattice plane of the deformed lattice overlap the physical space perfectly (Niizeki 1991a).

The previous theory of construction of a PA focuses on the point symmetry of the PA (Ishii 1989). Since there exist several Bravais classes with a given point symmetry,

0305-4470/91/204873+15$03.50

we have to determine the Bravais class to which the PA belongs (Niizeki 1991a, b). On account of this complication the theory is inconvenient as the basis of the present theory. Therefore, we will reformulate the theory focusing on the Bravais lattice of the PA (cf Verger-Gaugry 1988).

The theory of PAs has been developed in DMO along similar lines to the present one although the symmetry aspect of the PAs is only briefly considered. However, their method of obtaining PAs does not use deformed mother lattices but strips which are not parallel to the physical space, so that it does not fit into the theory of the space groups of the PAs (Niizeki 1991a). They confined, furthermore, to the case where the mother lattice of the QL is a simple hypercubic lattice. Their theory shares, nevertheless, several important points with our theory.

We investigate in section 2 the properties of the lattice planes of a higher-dimensional lattice. We investigate in section 3 the mother lattice of a QL and in section 4 those of the PAs to the QL. In section 5, a QL is constructed by the cut-and-projection method from its mother lattice and its self-similarity is investigated. The PAs to the QL are constructed in the same section from their mother lattices. We show also that the PAs are grouped into different series so that each series is generated from a prototype approximant by a successive application of the deflation-and-rescaling. In section 6 we apply the theory to the case of the octagonal QL. Sections 4 and 5 will be more easily understood if they are read in parallel with this section. Section 7 is devoted to discussions.

2. Lattice planes of a higher-dimensional lattice

Let E_D be the D-dimensional Euclidean space and $L(\subset E_D)$ a D-dimensional Bravais lattice. Then it forms a $\boldsymbol{Z}$-module (an additive group); $\boldsymbol{a}_1, \boldsymbol{a}_2 \in L \rightarrow n_1\boldsymbol{a}_1 + n_2\boldsymbol{a}_2 \in L$ $\forall (n_1, n_2) \in \boldsymbol{Z}^2$. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p \in L$ with $1 \leqslant p \leqslant D$ and assume that they are linearly independent over $\boldsymbol{R}$. Then they span a p-dimensional subspace Π_p of E_D and

$$\boldsymbol{Z}[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p] \equiv \left\{ \sum_i n_i\boldsymbol{a}_i \,\middle|\, (n_1, n_2, \ldots, n_p) \in \boldsymbol{Z}^p \right\} \qquad \left(\subset \Pi_p \right)$$

is a p-dimensional Bravais lattice generated by $\boldsymbol{a}_i$. Since $\boldsymbol{Z}[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p] \subset L \cap \Pi_p$, *$\Pi_p$ is a p-dimensional lattice plane of L.* More precisely, Π_1 is a lattice direction and $\Pi_D = E_D$. We shall call $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p\}$ a maximal set if $\boldsymbol{Z}[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p]$ is equal to $L_p \equiv L \cap \Pi_p$. A maximal set is a basis set of L_p, while $\boldsymbol{Z}[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p]$ in a non-maximal case is a superlattice (sublattice) of L_p.

If $\{\boldsymbol{a}_1\}$ is a maximal set, we shall call $\boldsymbol{a}_1$ a prime lattice vector because $\boldsymbol{a}_1/n \notin L$ for any integer n (>1). All the members of a maximal set must be prime vectors.

We may say that two sets of linearly independent vectors of L are equivalent to each other if both generate an identical lattice. Let $\{\boldsymbol{a}'_1, \boldsymbol{a}'_2, \ldots, \boldsymbol{a}'_p\}$ be another set of linearly independent vectors. Then this set is equivalent to the original one if $(\boldsymbol{a}_1\boldsymbol{a}_2\ldots\boldsymbol{a}_p) = (\boldsymbol{a}'_1\boldsymbol{a}'_2\ldots\boldsymbol{a}'_p)\mathbf{M}$ with $\mathbf{M} \in \mathrm{GL}_p(\boldsymbol{Z})$, where $(\boldsymbol{a}_1\boldsymbol{a}_2\ldots\boldsymbol{a}_p)$, for example, is a $D \times p$ matrix obtained by juxtaposing $\boldsymbol{a}_i$.

If $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p\}$ is not a maximal set, we may write $(\boldsymbol{a}_1\boldsymbol{a}_2\ldots\boldsymbol{a}_p) = (\boldsymbol{a}'_1\boldsymbol{a}'_2\ldots\boldsymbol{a}'_p)\mathbf{M}$ with $\mathbf{M}$ being a $p \times p$ integer matrix and $\{\boldsymbol{a}'_1, \boldsymbol{a}'_2, \ldots, \boldsymbol{a}'_p\}$ a basis set of L_p. Then $|\det(\mathbf{M})|$ (>1) *represents the number of the lattice points of L_p in a unit cell of $\boldsymbol{Z}[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p]$.* The set of primed vectors is called a reduced form of the non-maximal set.

Let $\{a_1, a_2, \ldots, a_p\}$ be a maximal set generating a lattice L_p. Then its every subset, e.g., $\{a_1, a_2, \ldots, a_q\}$ $(q<p)$, is maximal, too. A necessary and sufficient condition for $\{a_1, a_2, \ldots, a_p\}$ to be a maximal set is that there exists another maximal set $\{a_{p+1}, a_{p+2}, \ldots, a_D\}$ such that $\{a_1, a_2, \ldots, a_D\}$ is a basis set of L. We may say that the two sets $\{a_1, a_2, \ldots, a_p\}$ and $\{a_{p+1}, a_{p+2}, \ldots, a_D\}$ are complementary to each other and so are the two lattices L_p and $L_{D-p} \equiv Z[a_{p+1}, a_{p+2}, \ldots, a_D]$; $L = L_p + L_{D-p} (= \{l_1 + l_2 | l_1 \in L_p,\ l_2 \in L_{D-p}\})$. Note that L_{D-p} is not uniquely determined by L_p.

We consider at the moment the case $L = Z^D$, which is composed of integer vectors. $a \in Z^D$ is a prime vector if the greatest common measure among its components is trivial. Let $a_1, a_2, \ldots, a_p \in Z^D$ and assume that they are linearly independent. Then $\mathbf{K} \equiv (a_1 a_2 \ldots a_p)$ is a $D \times p$ integer matrix whose rank is equal to p. $\{a_1, a_2, \ldots, a_p\}$ with $p = D$ is maximal if $\mathbf{K}$ is unimodular. Therefore, we say that $\mathbf{K}$ with $1 \leq p < D$ is unimodular, too, if $\{a_1, a_2, \ldots, a_p\}$ is a maximal set. This is a natural generalization of unimodularity to the case of a rectangular matrix. The set of all $D \times p$ unimodular matrices is denoted by $\mathrm{Um}(D, p)$ $(\mathrm{Um}(D, D) = \mathrm{GL}_D(Z))$.

A necessary and sufficient condition for $\mathbf{K}$ to be unimodular is that there exists a $D \times (D-p)$ unimodular matrix $\mathbf{K}'$ such that $\mathbf{K} \cup \mathbf{K}' \in \mathrm{GL}_D(Z)$, i.e. $\mathbf{K}$ is embedded into a conventional unimodular matrix. We may say that $\mathbf{K}$ and $\mathbf{K}'$ are complementary to each other. Note that $\mathbf{K}'$ is not uniquely determined by $\mathbf{K}$.

If $\mathbf{K} \in \mathrm{Um}(D, p)$, then so are $\mathbf{KM}$ and $\mathbf{M}'\mathbf{K}$ with $\mathbf{M} \in \mathrm{GL}_p(Z)$ and $\mathbf{M}' \in \mathrm{GL}_D(Z)$. We shall define that $\mathbf{K}$, $\mathbf{K}' \in \mathrm{Um}(D, p)$ are equivalent if $\mathbf{K} = \mathbf{K}'\mathbf{M}$ with $\mathbf{M} \in \mathrm{GL}_p(Z)$. In particular, $\mathbf{K}$ and $-\mathbf{K}$ are equivalent. On the other hand, if $\mathbf{K}$ is a $D \times p$ integer matrix but not unimodular, it is decomposed as $\mathbf{K}'\mathbf{M}$ where $\mathbf{K}' \in \mathrm{Um}(D, p)$ and $\mathbf{M}$ is a $p \times p$ integer matrix. If the rank of $\mathbf{K}$ is p, we obtain $|\det(\mathbf{M})| > 1$. Then $\mathbf{K}'$ is called a reduced form of $\mathbf{K}$. $\mathbf{K}'$ is not uniquely determined by $\mathbf{K}$ but its equivalence class is.

According to the theory of elementary divisors of an integer matrix (see an appropriate textbook of algebra), there exists an algorithm for the above-mentioned decomposition: $\mathbf{K} = \mathbf{K}'\mathbf{M}$. The theory tells us also that $\mathbf{K}$ is unimodular if and only if the determinants of all the p-dimensional minors of $\mathbf{K}$ have no other common measures than ± 1. In particular, $\mathbf{K}$ is unimodular if one of the determinants is equal to 1 or -1.

We define $\mathrm{Um}(p, D) \equiv \{{}^t\mathbf{K} | \mathbf{K} \in \mathrm{Um}(D, p)\}$. $\mathbf{J}, \mathbf{J}' \in \mathrm{Um}(p, D)$ are equivalent if $\mathbf{J} = \mathbf{MJ}'$ with $\mathbf{M} \in \mathrm{GL}_p(Z)$. If $\mathbf{J} \in \mathrm{Um}(p, D)$, then it is embedded into a $p \times D$ block of $\mathbf{M} \in \mathrm{GL}_D(Z)$, so that we can conclude that $\{\mathbf{J}n | n \in Z^D\} = Z^p$ because $\{\mathbf{M}n | n \in Z^D\} = Z^D$. That is, $\mathbf{J}$ represents a surjection from Z^D onto Z^p. Conversely, if a $p \times D$ integer matrix $\mathbf{J}$ has this property, it is unimodular.

If $\mathbf{K} \in \mathrm{Um}(D, p)$ $(1 \leq p < D)$, there exists $\mathbf{J} \in \mathrm{Um}(D-p, D)$ such that $\mathbf{JK} = 0$. $\mathbf{J}$ is called a dual unimodular matrix to $\mathbf{K}$ and is denoted as $\mathbf{K}^{\perp}$. $\mathbf{K}^{\perp}$ is not uniquely determined by $\mathbf{K}$ but its equivalence class is.

We will return to the general case where L is spanned by a basis set $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_D\}$; $L = \{\sum_i n_i \varepsilon_i | (n_1, n_2, \ldots, n_D) \in Z^D\}$. Let Π_p be a lattice plane of L and $\{a_1, a_2, \ldots, a_p\}$ a basis set of $L_p \equiv L \cap \Pi_p$. Then, there exists $\mathbf{K} \in \mathrm{Um}(D, p)$ such that $(a_1 a_2 \ldots a_p) = (\varepsilon_1 \varepsilon_2 \ldots \varepsilon_D)\mathbf{K}$ and we can index Π_p by $\mathbf{K}$, which is equivalent to a (pD)-dimensional integer vector. In fact, $\mathbf{K}$ is not uniquely determined by Π_p but its equivalence class is.

A lattice direction Π_1 is indexed by a column integer vector $\mathbf{K}$, which is represented as $[k_1 k_2 \ldots k_D]$. More generally, $x = \sum_i h_i \varepsilon_i \in E_D$ is indexed as $[h_1 h_2 \ldots h_D]$. The index $\mathbf{K}$ of a 2D lattice plane Π_2 will be represented with brackets as $[n_1 n_2 \ldots n_D / m_1 m_2 \ldots m_D]$, where the first (or last) half of the integers show the first (or second) column of $\mathbf{K}$.

Let $\mathbf{J}=\mathbf{K}^{\perp}$. Then, Π_p can be indexed also by $\mathbf{J}$. This index scheme is called the dual index scheme to the one by $\mathbf{K}$. It is is a generalization of the Miller index used in indexing a lattice plane of a three-dimensional lattice. It is useful when $D-p$, the codimension of the lattice plane, is smaller than p. In this index scheme a hyperlattice plane Π_{D-1} is indexed by a row integer vector $\mathbf{J}$, which is represented as $(j_1 j_2 \ldots j_D)$. Similarly, the dual index $\mathbf{J}$ of a $(D-2)$-lattice plane Π_{D-2} will be represented with parentheses as $(n_1 n_2 \ldots n_D / m_1 m_2 \ldots m_D)$.

Let σ be a linear transformation which leaves L invariant. Then $\sigma(\boldsymbol{\varepsilon}_1 \boldsymbol{\varepsilon}_2 \ldots \boldsymbol{\varepsilon}_D) = (\boldsymbol{\varepsilon}_1 \boldsymbol{\varepsilon}_2 \ldots \boldsymbol{\varepsilon}_D)\mathbf{M}$ with $\mathbf{M} \in \mathrm{GL}_D(\boldsymbol{Z})$. A lattice plane Π_p indexed by $\mathbf{K}$ is transformed by σ to another one $\Pi'_p = \sigma\Pi_p$ and the index of Π'_p is given by $\mathbf{K}' = \mathbf{MK}$. Π_p is invariant against σ if there exists $\mathbf{M}' \in \mathrm{GL}_p(\boldsymbol{Z})$ such that $\mathbf{MK} = \mathbf{KM}'$, i.e. $\mathbf{K}'$ is equivalent to $\mathbf{K}$. This equation is a different expression of the equation $\sigma(\boldsymbol{a}_1 \boldsymbol{a}_2 \ldots \boldsymbol{a}_p) = (\boldsymbol{a}_1 \boldsymbol{a}_2 \ldots \boldsymbol{a}_p)\mathbf{M}'$ with $(\boldsymbol{a}_1 \boldsymbol{a}_2 \ldots \boldsymbol{a}_p) = (\boldsymbol{\varepsilon}_1 \boldsymbol{\varepsilon}_2 \ldots \boldsymbol{\varepsilon}_D)\mathbf{K}$.

3. The mother lattice of a quasilattice

We take a non-crystallographic point group G in d-dimensions with $d=2$ or 3 and assume that it has a faithful unimodular representation whose dimension is equal to $D=2d$. The representation is equivalent to a D-dimensional point group $\hat{\mathrm{G}}$, which is a finite subgroup of the D-dimensional orthogonal group. $\hat{\mathrm{G}}$ is isomorphic to G. The Euclidean space E_D onto which $\hat{\mathrm{G}}$ acts is decomposed into two d-dimensional invariant subspaces, $E_D = E_d \oplus E'_d$, and the restriction of $\hat{\mathrm{G}}$ onto E_d is identical to G. We shall call E_d the physical space and E'_d the internal one.

Let G′ be the restriction of $\hat{\mathrm{G}}$ onto E'_d. Then it is the same d-dimensional point group as G. The two bijections $\mathrm{G} \leftarrow \hat{\mathrm{G}} \rightarrow \mathrm{G}'$ induce a bijection $\varphi: \mathrm{G} \rightarrow \mathrm{G}'$, which is an isomorphism. However, φ is not isomorphism as d-dimensional point groups; for example, the rotation through $2\pi/5$ in the case of G = 10mm is mapped by φ onto the rotation through $4\pi/5$ in G′.

Let $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \ldots, \boldsymbol{\varepsilon}_D$ be the basis vectors of the unimodular representation of G. Then the D-dimensional lattice

$$L = \left\{ \sum_i n_i \boldsymbol{\varepsilon}_i \,\middle|\, (n_1, n_2, \ldots, n_D) \in \boldsymbol{Z}^D \right\} \tag{1}$$

is a Bravais lattice which is invariant against $\hat{\mathrm{G}}$. We assume that $\hat{\mathrm{G}}$ is the point group of L. Then G and G′ as well as $\hat{\mathrm{G}}$ include the inversion operation. The space group of L is given by $\mathcal{g} = \{\{\sigma | \boldsymbol{l}\} | \sigma \in \hat{\mathrm{G}}, \boldsymbol{l} \in L\}$ ($\equiv \hat{\mathrm{G}} * L$, the semidirect product).

Let $\boldsymbol{x} \in E_D$. Then $\mathcal{g}(\boldsymbol{x}) \equiv \{\alpha | \alpha \in \mathcal{g}, \alpha\boldsymbol{x} = \boldsymbol{x}\}$ represents the point symmetry of $\boldsymbol{x}$ with respect to L. $\hat{\mathrm{G}}(\boldsymbol{x}) \equiv \{\sigma | \{\sigma | \boldsymbol{l}\} \in \mathcal{g}(\boldsymbol{x})\}$ is a subgroup of $\hat{\mathrm{G}}$ and is called the point group of $\boldsymbol{x}$. $\boldsymbol{x}$ is called a special point if $\hat{\mathrm{G}}(\boldsymbol{x})$ is a centring group.

The point groups which fit the above considerations are restricted to 8mm (D_8), 10mm (D_{10}) and 12mm (D_{12}) if $d=2$ and $\mathrm{m}\bar{3}\bar{5}$ (Y_h) if $d=3$ (Janssen 1988). In the case of $d=2$, there exists only one Bravais class for each point group. That is, we have three four-dimensional (4D) Bravais lattices, p8mm, p10mm and p12mm. On the other hand, there exist three 6D lattices, $\mathrm{Pm}\bar{3}\bar{5}$, $\mathrm{Fm}\bar{3}\bar{5}$ and $\mathrm{Im}\bar{3}\bar{5}$ for the case of $d=3$.

Let P and P' be the projectors onto E_d and E'_d, respectively. Then $\boldsymbol{e}_i = P\boldsymbol{\varepsilon}_i$ (or $\boldsymbol{e}'_i = P'\boldsymbol{\varepsilon}_i$) are linearly independent over $\boldsymbol{Z}$ and the $\boldsymbol{Z}$-module $L_P \equiv PL = \{\Sigma_i n_i \boldsymbol{e}_i | (n_1, n_2, \ldots, n_D) \in \boldsymbol{Z}^D\}$ (or $L'_P \equiv P'L$) is a dense set in E_d (or E'_d) and called a pre-quasilattice. $\boldsymbol{e}_i$ are subject to a unimodular transformation by the action of G and L_p is invariant against G. If $\boldsymbol{l}=\Sigma_i n_i \boldsymbol{\varepsilon}_i \in L$, then $P\boldsymbol{l}=\Sigma_i n_i \boldsymbol{e}_i$ and $P'\boldsymbol{l}=\Sigma_i n_i \boldsymbol{e}'_i$; $\boldsymbol{\varepsilon}_i=(\boldsymbol{e}_i, \boldsymbol{e}'_i)$. E_d has an incommensurate orientation with respect to L and $L\cap E_d=\{0\}$.

Let $\tau=1+\sqrt{2}$, $2+\sqrt{3}$ or $2+\sqrt{5}$ for p8mm, p12mm and Pm$\bar{3}\bar{5}$, respectively, but $\tau=(1+\sqrt{5})/2$ for p10mm, Fm$\bar{3}\bar{5}$ and Im$\bar{3}\bar{5}$. Then the D-dimensional linear transformation $\hat{\tau}=\tau\mathbf{I}\oplus\tau'\mathbf{I}$ with $\mathbf{I}$ being a d-dimensional unit matrix and τ' the algebraic conjugate of τ induces a unimodular transformation among $\boldsymbol{\varepsilon}_i$:

$$\hat{\tau}(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2\ldots\boldsymbol{\varepsilon}_D)=(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2\ldots\boldsymbol{\varepsilon}_D)\mathbf{M} \tag{2}$$

where $\mathbf{M}\in \mathrm{GL}_D(\boldsymbol{Z})$. It follows that $\hat{\tau}L=L$. $\hat{\tau}$ acts as a scale transformation onto each of the two subspaces E_d and E'_d and is commutable with $\hat{\mathrm{G}}$. Note that the row vectors of $(\boldsymbol{e}_1\boldsymbol{e}_2\ldots\boldsymbol{e}_D)$ (or $(\boldsymbol{e}'_1\boldsymbol{e}'_2\ldots\boldsymbol{e}'_D)$) are left eigenvectors of $\mathbf{M}$ with respect to its eigenvalue τ (or τ') and $|\tau'|=1/\tau$ (<1).

We exclude hereafter the case of p12mm from our considerations. Then τ satisfies the quadratic equation $\tau^2=m\tau+1$, where $m=1$, 2 or 4 according as $\tau=(1+\sqrt{5})/2$, $1+\sqrt{2}$ or $2+\sqrt{5}$, respectively. It follows that $\tau'=-1/\tau$. The Fibonacci numbers or their analogues are defined by the recursion relation $u_{k+1}=mu_k+u_{k-1}$ with $u_0=0$ and $u_1=1$. u_{k+1}/u_k is a best approximant to τ. From $\tau^2=m\tau+1$ we obtain $\tau^k=u_k\tau+u_{k-1}$. Accordingly $\mathbf{M}^2=m\mathbf{M}+\mathbf{I}$ and $\mathbf{M}^k=u_k\mathbf{M}+u_{k-1}\mathbf{I}$.

Let Π_d be a d-dimensional lattice plane of L and assume that $\mathbf{K}\in \mathrm{Um}(D, d)$ is its index. Then the slope of the transformed lattice plane $\Pi'_d\equiv\hat{\tau}\Pi_d$ relative to E_d is smaller than that of Π_d because $\hat{\tau}$ enlarges E_d but shrinks E'_d. If $\hat{\tau}$ is operated successively onto Π_d, we obtain a series of lattice planes, $\Pi_d^{(k)}=(\hat{\tau})^k\Pi_d$, $k=0, 1, 2, \ldots$, which tend to E_d (DMO). $\Pi_d^{(k)}$ is indexed by $\mathbf{K}_k=\mathbf{M}^k\mathbf{K}=u_k\mathbf{K}'+u_{k-1}\mathbf{K}$ with $\mathbf{K}'=\mathbf{KM}$ $(=\mathbf{K}_1)$.

Let $\mathbf{J}$ and $\mathbf{J}_k$ be the dual indices to $\mathbf{K}$ and $\mathbf{K}_k$, respectively. Then we obtain $\mathbf{J}_k=\mathbf{J}(-\mathbf{M}^{-1})^k=u_k\mathbf{J}'+u_{k-1}\mathbf{J}$ with $\mathbf{J}'=-\mathbf{J}\mathbf{M}^{-1}$ because $-\mathbf{M}^{-1}$ satisfies the same quadratic equation as $\mathbf{M}$.

The point group of Π_d is defined by the maximal subgroup of $\hat{\mathrm{G}}$ among those which leave Π_d invariant. The point group is common among $\Pi_d^{(k)}$ because $\hat{\tau}$ is commutable with $\hat{\mathrm{G}}$.

4. The mother lattices of periodic approximants to a quasilattice

Let us deform L by introducing a linear phason strain so that its lattice plane Π_d coincides with the physical space E_d. That is, E_d is a lattice plane of the deformed lattice $\tilde{L}$ and E_d is fully commensurate with $\tilde{L}$. We may write $\Phi\Pi_d=E_d$ and $\tilde{L}=\Phi L$ with Φ being a D-dimensional transformation matrix associated with the phason strain. Φ is divided into four blocks as

$$\Phi=\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{S} & \mathbf{I}\end{pmatrix} \tag{3}$$

where $\mathbf{S}$ is a $d\times d$ matrix representing the phason strain (Niizeki 1991a). Φ acts onto E_d as the identity transformation. Note that $\det(\Phi)=1$, so that the transformation is volume-conserving.

Let $\tilde{\boldsymbol{\varepsilon}}_i=\Phi\boldsymbol{\varepsilon}_i$. Then we obtain

$$\tilde{L}=\left\{\sum_i n_i\tilde{\boldsymbol{\varepsilon}}_i \,\middle|\, (n_1, n_2, \ldots, n_D)\in \boldsymbol{Z}^D\right\}. \tag{4}$$

Only the internal components of ε_i are changed by Φ; $\tilde{\varepsilon}_i = (e_i, \tilde{e}'_i)$ with $\tilde{e}'_i = e'_i + \mathbf{S}e_i$ or, equivalently,

$$(\tilde{e}'_1 \tilde{e}'_2 \ldots \tilde{e}'_D) = (e'_1 e'_2 \ldots e'_D) + \mathbf{S}(e_1 e_2 \ldots e_D). \tag{5}$$

Let $\mathbf{K} \in \mathrm{Um}(D, p)$ be the index of Π_d. Then the column vectors of $(\varepsilon_1 \varepsilon_2 \ldots \varepsilon_D)\mathbf{K}$ span Π_d. It follows that $P'\{(\tilde{\varepsilon}_1 \tilde{\varepsilon}_2 \ldots \tilde{\varepsilon}_D)\mathbf{K}\} = 0$ because $\Phi\Pi_d = E_d$ and, consequently,

$$(\tilde{e}'_1 \tilde{e}'_2 \ldots \tilde{e}'_D)\mathbf{K} = 0. \tag{6}$$

Let $(a_1 a_2 \ldots a_d) \equiv P\{(\tilde{\varepsilon}_1 \tilde{\varepsilon}_2 \ldots \tilde{\varepsilon}_D)\mathbf{K}\}$. Then we obtain

$$(a_1 a_2 \ldots a_d) = (e_1 e_2 \ldots e_D)\mathbf{K}. \tag{7}$$

Inserting (5) into (6) yields $\mathbf{A}' + \mathbf{SA} = 0$ with $\mathbf{A} \equiv (a_1 a_2 \ldots a_d)$ and $\mathbf{A}' \equiv (e'_1 e'_2 \ldots e'_D)\mathbf{K}$, so that $\mathbf{S} = -\mathbf{A}'\mathbf{A}^{-1}$. Consequently, $\tilde{L}$ as well as $\mathbf{S}$ is determined if Π_d (or K) is specified. Therefore $\tilde{L}$ may be indexed by $\mathbf{K}$.

$\Phi^{(1)} \equiv \sigma\Phi\sigma^{-1}$ with $\sigma \in \hat{\mathrm{G}}$ takes a similar form as (3) because σ decomposes into two point groups acting on E_d and E'_d. Assume moreover that $\sigma\Pi_d = \Pi_d$. Then $\Phi^{(1)}$ as well as Φ transforms Π_d to E_d, so that $\Phi^{(1)} = \Phi$ on account of the uniqueness. Therefore Φ is commutable with $\hat{\mathrm{H}} \equiv \{\sigma | \sigma \in \hat{\mathrm{G}}, \sigma\Pi_d = \Pi_d\}$, which is nothing but the point group of Π_d. $\hat{\mathrm{H}}$ acts on E_d (or E'_d) as a d-dimensional point group H (or H′) and φ represents a bijection from H onto H′. In fact, H and H′ are identical and crystallographic in d dimensions. Therefore S must commute with H and it takes the form $S = \sum_i \lambda_i P_i$, where λ_i are constants and P_i the projectors onto the invariant subspaces of H (cf Ishii 1989). In particular, $S = \lambda I$ if H is irreducible.

The space group of $\tilde{L}$ is given by $\tilde{\mathscr{g}} = \hat{\mathrm{H}} * \tilde{L}$. Note that H and H′ as well as $\hat{\mathrm{H}}$ include the inversion operation.

When L is deformed to $\tilde{L}$, $x \in E_D$ is transformed to $\tilde{x} = \Phi x$. The point group $\hat{\mathrm{H}}(\tilde{x})$ of $\tilde{x}$ with respect to $\tilde{L}$ is related to $\hat{\mathrm{G}}(x)$ by $\hat{\mathrm{H}}(\tilde{x}) = \hat{\mathrm{H}} \cap \hat{\mathrm{G}}(x)$, which is a subgroup of $\hat{\mathrm{G}}(x)$. Every special point of $\tilde{L}$ has its associate among the special points of L.

The d-dimensional lattice $L_d \equiv \tilde{L} \cap E_d$ is given by

$$L_d = \left\{ \sum_i n_i a_i \,\middle|\, (n_1, n_2, \ldots, n_d) \in \mathbf{Z}^d \right\}. \tag{8}$$

Its point group is given by H and its space group by $\mathscr{g}_d = \mathrm{H} * L_d$. It is obvious that $L_d = P(L \cap \Pi_d)$. The point group H of L_d is determined more easily than the point group $\hat{\mathrm{H}}$ of Π_d. $\hat{\mathrm{H}}$ can be obtained by lifting H up to the D-dimensional point group.

Equation (6) shows that only d of $\tilde{e}'_i$ are linearly independent over $\mathbf{Z}$, so that $\tilde{e}'_i$ are given as linear combinations of d-vectors with integer coefficients. In fact, we may write

$$(\tilde{e}'_1 \tilde{e}'_2 \ldots \tilde{e}'_D) = (b_1 b_2 \ldots b_d)\mathbf{J} \tag{9}$$

because of (6), where $b_i \in E'_d$ and $\mathbf{J} = \mathbf{K}^{\perp}$. Since $\tilde{e}'_i$ are projections of the basis vectors of $\tilde{L}$ onto E'_d, b_i must be linearly independent over $\mathbf{R}$. Equations (5) and (9) show that $\tilde{e}'_i$ are rational approximants to e'_i if S is small.

The projection of $\tilde{L}$ onto E'_d is called the shadow lattice of $\tilde{L}$ (Niizeki 1991a) and denoted by L_s; $L_s \equiv P'\tilde{L}$. Using (9) together with $\mathbf{J} \in \mathrm{Um}(D, d)$, we obtain

$$L_s = \left\{ \sum_i \bar{n}_i b_i \,\middle|\, (\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_d) \in \mathbf{Z}^d \right\} \tag{10}$$

so that b_i are basis vectors of L_s. The point group of L_s is given by H′ (= H) and its space group by $\mathscr{g}_s = \mathrm{H}' * L_s$. L_s does not necessarily belong to the same Bravais class as that of L_d though the point group is common.

The projection $L \to L_s$ is a surjection, which together with the bijection $L \to L_P$ yields another surjection: $\Sigma_i n_i \boldsymbol{e}_i \in L_P \to \Sigma_i n_i \tilde{\boldsymbol{e}}'_i \in L_s$. The latter surjection may be represented by the symbol φ which represents the bijection $\mathrm{H} \to \mathrm{H}'$. Then φ is extended naturally to a surjection from $\mathcal{g}_P \equiv \mathrm{H} * L_P$ onto $\mathcal{g}_s$. $\mathcal{g}_P$ is a quasi-space-group which is a subgroup of the maximal quasi-space-group, $\mathrm{G} * L_P$, of L_P. Note that L_d is the kernel of the surjection $\varphi : L_P \to L_s$.

We can write $L_s = P'L'_d$, where L'_d is a complementary lattice to L_d; $L = L_d + L'_d$. Note that $P'L'_d$ is determined uniquely in contrast to L'_d. L_d is called a base lattice and L'_d a fibre lattice by Sadoc and Mosseri (1990). L'_d and L_s are isomorphic as $\boldsymbol{Z}$-modules and the basis vectors of L'_d can be so chosen that their projections onto E'_d coincide with $\boldsymbol{b}_i$.

Let $\Phi_k \equiv (\hat{\tau})^k \Phi (\hat{\tau})^{-k}$. Then it is written as (3) but $\mathbf{S}$ is replaced by $\mathbf{S}_k = (-1/\tau^2)^k \mathbf{S}$, which represents a weaker phason strain than $\mathbf{S}$. The lattice plane $\Pi_d^{(k)} = (\hat{\tau})^k \Pi_d$ of L is transformed by Φ_k to E_d; $\Phi_k \Pi_d^{(k)} = E_d$ because $\Phi \Pi_d = E_d$ and $\hat{\tau} E_d = E_d$. Therefore E_d is a lattice plane of $\tilde{L}_k = \Phi_k L$. Using $\hat{\tau} L = L$ and $\Phi L = \tilde{L}$, we can rewrite as $\tilde{L}_k = (\hat{\tau})^k \tilde{L}$. We shall call $\tilde{L}_k$ the kth generation of deformed lattices; $\tilde{L}$ is the zeroth one. Two successive generations are related by $\hat{\tau}$ as $\hat{\tau} \tilde{L}_k = \tilde{L}_{k+1}$.

Since $\hat{\tau} = \tau \mathbf{I} \oplus \tau' \mathbf{I}$, we obtain $L_d^{(k)} \equiv \tilde{L}_k \cap E_d = \tau^k L_d$ (DMO), which is similar to L_d. Thereafter the basis vectors of $L_d^{(k)}$ are given by $\boldsymbol{a}_i^{(k)} = \tau^k \boldsymbol{a}_i$, $i = 1 - d$. Similarly, $L_s^{(k)}$, the shadow lattice of $\tilde{L}_k$, is equal to $\tau^{-k} L_s$, which is similar to L_s. The basis vectors of $L_s^{(k)}$ are given by $\boldsymbol{b}_i^{(k)} = \tau^{-k} \boldsymbol{b}_i$, $i = 1 - d$.

5. Construction of a quasilattice and its periodic approximants by the cut-and-projection method

5.1. The case of a quasilattice

A QL (quasilattice) is obtained from L by the cut-and-projection method as $Q(\boldsymbol{\phi}, W) = \{P\boldsymbol{l} \mid \boldsymbol{l} \in L,\ P'\boldsymbol{l} + \boldsymbol{\phi} \in W\}$, where $\boldsymbol{\phi} \in E'_d$ is the phase vector and W ($\subset E'_d$) the window. W is a polygonal (or polyhedral) domain which is invariant against G'; the origin of E'_d is the centre of the inversion symmetry of W. It is usual that $W = P'\Gamma$ with $\Gamma \subset E_D$ being a polytope (e.g. a Voronoi polytope), which we shall assume hereafter. Γ is invariant against $\hat{\mathrm{G}}$.

$Q (= Q(\boldsymbol{\phi}, W))$ is a discrete subset of L_P. It has quasiperiodicity and its macroscopic point symmetry is given by G. Two QLs with different phase vectors but a common window are locally isomorphic. The average density of the lattice points of Q is proportional to $\mathrm{vol}(W)$.

We now consider the self-similarity of a QL. We begin with the relation $Q(\boldsymbol{\phi}, W/\tau) \subsetneq Q(\boldsymbol{\phi}, W)$ because $W/\tau \subsetneq W$. Using $\hat{\tau} L = L$, we can show eaily that $\tau^{-1} Q(\boldsymbol{\phi}, W/\tau) = Q(-\tau\boldsymbol{\phi}, W)$ (Niizeki 1989a) and a subset of $Q(\boldsymbol{\phi}, W)$ is locally isomorphic to itself if it is rescaled. Similarly, $Q(\boldsymbol{\phi}, \tau W) \supsetneq Q(\boldsymbol{\phi}, W)$ and $\tau Q(\boldsymbol{\phi}, \tau W) = Q(-\boldsymbol{\phi}/\tau, W)$, which is locally isomorphic to $Q(\boldsymbol{\phi}, W)$. $Q(\boldsymbol{\phi}, W/\tau)$ (or $Q(\boldsymbol{\phi}, \tau W)$) is called an inflation (or deflation) of $Q(\boldsymbol{\phi}, W)$. The procedure for obtaining $\tau^{-1} Q(\boldsymbol{\phi}, W/\tau)$ (or $\tau Q(\boldsymbol{\phi}, \tau W)$) is called inflation-and-rescaling (or deflation-and-rescaling). The two procedures derive from a QL new QLs which are locally isomorphic to the original one; they are inverse to each other. A QL is self-similar in the sense that there exist these procedures.

We shall rewrite the expression for $Q(\boldsymbol{\phi}, W)$ slightly for a later convenience. There exists $\boldsymbol{x} \in E_D$ such that $\boldsymbol{\phi} = P'\boldsymbol{x}$. Then the shifted QL, $Q(\boldsymbol{x}, W) \equiv P\boldsymbol{x} + Q(\boldsymbol{\phi}, W)$, is written as

$$Q(x, W) = \{P(l+x) \mid l \in L,\ P'(l+x) \in W\}. \tag{11}$$

It can be shown easily that the deflation-and-rescaling (DAR) of $Q(x, W)$ is equal to $Q(\hat{\tau}x, W)$ (see also the appendix). More generally, its kth DAR is equal to $Q((\hat{\tau})^k x, W)$.

We consider here a special case where x is a special point of L. Then the origin is the centre of the global symmetry of $Q(x, W)$ (Niizeki 1989b); the point group is equal to $\mathrm{G}(x)$, the restriction of $\hat{\mathrm{G}}(x)$ onto E_d. $\hat{\mathrm{G}}(\hat{\tau}x) = \hat{\mathrm{G}}(x)$ but the special point $\hat{\tau}x$ may not be equivalent to x; the point symmetry is not changed by the DAR but the local pattern around the centre of symmetry may be changed (Niizeki 1989b). The initial QL is recovered after a finite number of DARs. The number is the smallest one among those satisfying $(\hat{\tau})^k x \equiv x \bmod L$.

5.2. *The case of periodic approximants*

A PA (periodic approximant) to the QL given by (11) is defined naturally (Niizeki 1991a) as

$$\tilde{Q}(\tilde{x}, \tilde{W}) = \{P(l+\tilde{x}) \mid l \in \tilde{L},\ P'(l+\tilde{x}) \in \tilde{W}\} \tag{12}$$

where $\tilde{x} = \Phi x$ and $\tilde{W} = P'\Phi\Gamma$. More generally, the kth generation of the PA is defined by

$$\tilde{Q}_k(\tilde{x}_k, \tilde{W}_k) = \{P(l+\tilde{x}_k) \mid l \in \tilde{L}_k,\ P'(l+\tilde{x}_k) \in \tilde{W}_k\} \tag{13}$$

with $\tilde{x}_k = (\hat{\tau})^k \tilde{x}$ and $\tilde{W}_k = P'\Phi_k\Gamma$. Here, the suffix k of $\tilde{Q}_k \equiv Q_k(\tilde{x}_k, \tilde{W}_k)$ means that it is obtained by the cut-and-projection method from $\tilde{L}_k$. $\tilde{W}_k$ is weakly deformed from W provided that the phason strain S_k is not too large, which we shall assume hereafter. The point symmetry of $\tilde{W}_k$ ($\tilde{W}_0 = \tilde{W}$) is given by H$'$. In particular, $\tilde{W}_k = -\tilde{W}_k$, i.e., $\tilde{W}_k$ has the inversion symmetry.

Let $\tilde{\phi} = P'\tilde{x}$ and $\mathscr{g}_s(\tilde{\phi}) = \{\alpha \mid \alpha \in \mathscr{g}_s,\ \alpha\tilde{\phi} = \tilde{\phi}\}$, i.e. the point group of $\tilde{\phi}$ with respect to L_s. Then the space group of $\tilde{Q}(\tilde{x}, \tilde{W})$ is given by $\tilde{\mathscr{g}}_P(\tilde{x}) = \varphi^{-1}(\mathscr{g}_s(\tilde{\phi}))$ (Niizeki 1991a), which is independent of $\tilde{W}$. The translational part of the space group is given by L_d because L_d is the kernel of φ.

$\tilde{Q}$ is called a regular PA if its point symmetry conforms to the Bravais lattice L_d. In order to obtain a regular PA, it is necessary that $\tilde{\phi}$ is located on a special point or a special line of L_s (Niizeki 1991a). On the other hand, if $\tilde{x}$ is a special point of $\tilde{L}$, $P\tilde{x}$ is a special point of $\tilde{Q}$ and its point group is equal to $H(\tilde{x})$, the restriction of $\hat{H}(\tilde{x})$ onto E_d (Niizeki 1991a).

It is important that we have defined $\tilde{x}_k$ by $\tilde{x}_k = (\hat{\tau})^k\tilde{x}$ but not by $\tilde{x}_k = \Phi_k x$. Since $\tilde{x}_k = \Phi_k(\hat{\tau})^k x$, $\tilde{Q}_k(\tilde{x}_k, \tilde{W}_k)$ is not a PA to $Q(x, W)$ but to $Q((\hat{\tau})^k x, W)$, i.e. the kth DAR of $Q(x, W)$. Owing to this definition, we can prove as given in the appendix that

$$\tilde{Q}_k(\tilde{x}_k, \tilde{W}_k) = \tau^k \tilde{Q}(\tilde{x}, \tau^k \tilde{W}_k). \tag{14}$$

It is interesting that $\tilde{Q}_k$ is obtained not only from $\tilde{L}_k$ by (13) but also from $\tilde{L}$ by (14) with (12) (cf (22) in DMO).

By the assumption that $\tilde{W}_k$ and $\tilde{W}_{k-1}$ are not strongly deformed from W, we may assume that $\tilde{W}_k \supsetneqq \tau^{-1}\tilde{W}_{k-1}$. Using this together with (12) and (14) we can easily prove that $\tilde{Q}_k \supsetneqq \tau\tilde{Q}_{k-1}$ (this is directly proved from (A1)). It follows that $\tilde{Q}_k$ is obtained from $\tilde{Q}_{k-1}$ by the DAR and, conversely, $\tilde{Q}_{k-1}$ is obtained from $\tilde{Q}_k$ by the inflation-and-rescaling. The series of PAs, $\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2, \ldots$, are generated from $\tilde{Q}$ ($= \tilde{Q}_0$) by a successive application of the DAR. In fact, (14) means that $\tilde{Q}_k$ is derived as the kth DAR of $\tilde{Q}$.

We cannot only descend a series of PAs by the DAR but also ascend it by the inflation-and-rescaling. One can ascend a step only when the window of the earlier generation includes τ^{-1} times that of the later one. Therefore, the series terminates in a finite step in the ascending direction becuase the deformation of the window becomes increasingly large. We shall call the terminal PA a prototype of the series because the series starts from it and is descended by successive applications of the DAR. We can assume that $\tilde{Q}$ $(=\tilde{Q}_0)$ is the prototype PA.

The two QLs, $\tilde{Q}(\tilde{x}, \tau^k \tilde{W}_k)$ and $\tilde{Q}(\tilde{x}, \tilde{W})$, have a common phase vector and they have the same space group. Thus, we have arrived at the main conclusion of the present paper: PAs to a QL are grouped into series so that each series is generated from their prototype by a successive application of the DAR and the space group is common among the members of the series.

6. Application of the theory to the octagonal quasilattice

An octagonal QL is obtained from the 4D octagonal lattice $L = \mathrm{p8mm}$. Let $\boldsymbol{\varepsilon}_i = (\boldsymbol{e}_i, \boldsymbol{e}'_i)$ with $\boldsymbol{e}_i \in E_2$ and $\boldsymbol{e}'_i \in E'_2$ be the basis vectors of L. Then $\boldsymbol{e}_i$ (or $\boldsymbol{e}'_i$) are related to each other by $\boldsymbol{e}_{i+1} = r\boldsymbol{e}_i$ (or $\boldsymbol{e}'_{i+1} = r'\boldsymbol{e}'_i$) with $i = 2, 3$ and 4, where r (or r') is the rotation through $\pi/4$ (or $-3\pi/4$). It follows that $|\boldsymbol{e}_i|$ (or $|\boldsymbol{e}'_i|$) take a common value, which we shall denote by a (or a'). Note that $\boldsymbol{e}_i$ and $\boldsymbol{e}_{i+2}$ are perpendicular to each other and so are $\boldsymbol{e}'_i$ and $\boldsymbol{e}'_{i+2}$.

Eight vectors $\pm\boldsymbol{e}_i$ (or $\pm\boldsymbol{e}'_i$), $i = 1$–4, represent the vertex vectors of a regular octagon, whose point group is 8mm. The 4D rotation $\hat{r} \equiv r \oplus r'$ is an element of the point group $\hat{\mathrm{G}}$ ($\simeq$8mm) of L. $\hat{\mathrm{G}}$ is generated by $\hat{r}$ and the 4D mirror $\hat{\sigma}$ which transforms $\boldsymbol{\varepsilon}_i$ as $\hat{\sigma}(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_4) = (\boldsymbol{\varepsilon}_4, \boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_1)$. $\hat{\sigma}$ acts onto E_2 and E'_2 as 2D mirrors σ and σ'; $\hat{\sigma} = \sigma \oplus \sigma'$.

We can rescale E'_2 so that $a' = a$. Then L coincides with a simple hypercubic lattice. Let Γ be the Voronoi cell of the origin of L. Then $P'\Gamma$ is a regular octagon, which is the canonical window of the octagonal QL. The octagonal QL, $Q(x, W)$ with $W = P'\Gamma$, is formed of the vertices of the Ammann octagonal quasiperiodic tiling as shown in figure 1.

L has six classes of special points (Niizeki 1989b, 1990). The six are represented by the symbols, Γ, X, C, M, R and O, representatives of which are [0000], [h000], [hh00], [h0h0], [0hhh] and [$hhhh$] with $h = 1/2$, respectively. The point groups of Γ and O are 8mm, those of X, C and R are mm and that of M is 4mm. The vertices of the octagonal tiling in figure 1 are derived from Γ, the mid-ponts of the bonds from X and the centres of rhombi (or squares) from C (or M).

The octagonal QL has self-similarity with the scale $\tau = 1 + \sqrt{2}$ a shown in figure 1. The unimodular matrix associated with $\hat{\tau} = \tau\mathbf{l} \oplus \tau'\mathbf{l}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}. \tag{15}$$

The eight mirrors of 8mm are grouped into two classes, Σ and Δ; a mirror of type Σ passes the mid-point of an edge of the regular octagon formed by $\pm\boldsymbol{e}_i$, while the one of type Δ passes a vertex. A representative of Σ is σ and that of Δ is $r\sigma$.

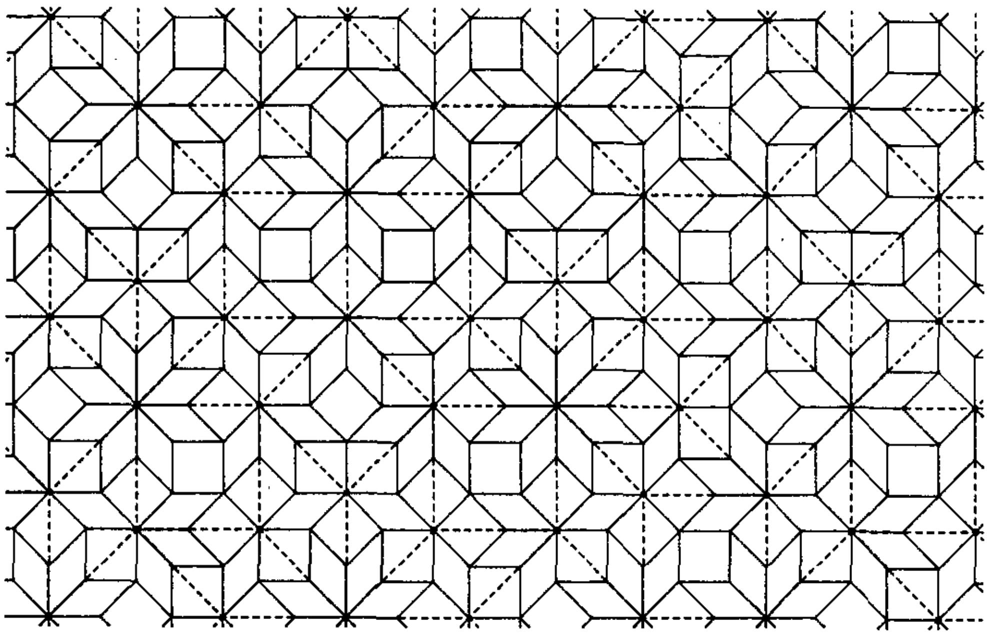

Figure 1. The octagonal quasilattice obtained from the 4D octagonal lattice L by using the canonical window. The lattice points are given by the positions of the vertices of the octagonal Ammann tiling. The centres of squares (or rhombi) are derived from the special points of type M (or C) of L. The once inflated QL is superimposed with the dashed lines. A tile of the inflated QL share a common centre with a similar tile of the original QL; both tiles are of the same kind but the orientations can be different.

The series of Fibonacci number analogues associated with τ $(=1+\sqrt{2})$ is given by $\{u_k\}=\{0, 1, 2, 5, 12, 29, \ldots\}$, in which the parity alternates. Let $v_k \equiv u_{k+1}-u_k$ $(=u_k-u_{k-1})$. Then, $\{v_k\}=\{1, 1, 3, 7, 17, \ldots\}$ is a series of odd numbers and generated by the same recursion relation as that of u_k. Note that v_k/u_k is a best approximant to $\sqrt{2}$ and $\tau^k = v_k+\sqrt{2}u_k$.

A PA (periodic approximant) to the octagonal QL yields a periodic tiling with the same tiles as the octagonal quasiperiodic tiling. The area of a unit cell of a PA is written as $\Omega = N_S\Omega_S + N_R\Omega_R$, where $\Omega_S = a^2$ (or $\Omega_R = a^2/\sqrt{2}$) is the area of the square (or rhombic) tile and N_S (or N_R) the number of the square (or rhombic) tiles in the unit cell. The total number of the tiles is given by $N = N_S + N_R$, which is equal also to the number of the lattice points in the unit cell. N increases in a series of approximants following the recursion relation $N_{k+1} = 6N_k - N_{k-1}$. N_S and N_R increase in the same way.

We shall investigate square approximants (p4mm) and rhombic ones (cmm). These approximants have two mirrors perpendicular to each other. Such mirrors in a PA must be of the same type (Σ or Δ). Therefore there exist four cases: (A) p4mm of type Σ; (B) p4mm of type Δ; (C) cmm of type Σ; and (D) cmm of type Δ.

Let us take the cartesian coordinate systems for E_2 and E_2' so that the two axes coincide with the two mirrors. Then the phason strain must be a diagonal matrix; $S_{12} = S_{21} = 0$. In the case of a square approximant, we obtain $S_{11} = S_{22}$ and $S = S_{11}I$.

Since all the special points (SPS) of L have the inversion symmetry,they remain as SPS after the phason strain is introduced. The SPS of classes X, M and R have mirrors of type Δ only, while those of type C have mirrors of type Σ only. The mirrors of an SP are lost by the introduction of the phason strain if they do not conform of the type (Σ or Δ) of the strain.

A series of deformed lattices, $\tilde{L}_k$, $k=0,1,2,\ldots$, is characterized by the index $\mathbf{K}$ of the prototype $\tilde{L}$ $(=\tilde{L}_0)$. We choose a most important series from each of the four cases. The index $\mathbf{K}$ and its dual $\mathbf{J}$ are listed in table 1. The index $\mathbf{K}_k$ of the kth generation and its dual $\mathbf{J}_k$ are also listed.

The phason strain S and the basis vectors $\langle \boldsymbol{a}_1, \boldsymbol{a}_2\rangle$ of L_2 are as follows:

(A)	$S=\tau^{-1}I$		$\langle \boldsymbol{e}_2+\boldsymbol{e}_3, -\boldsymbol{e}_1+\boldsymbol{e}_4\rangle$
(B)	$S=-I$		$\langle \boldsymbol{e}_1, \boldsymbol{e}_3\rangle$
(C)	$S_{11}=\tau^{-1}$	$S_{22}=-\tau$	$\langle \boldsymbol{e}_2, \boldsymbol{e}_3\rangle$
(D)	$S_{11}=-\tau^{-2}$	$S_{22}=1$	$\langle \boldsymbol{e}_1-\boldsymbol{e}_4, \boldsymbol{e}_1+\boldsymbol{e}_2\rangle$.

L_s belongs to the same Bravais class as that of L_2 except the case (D), where L_s belongs to pmm. The unit cells are 45°-rhombi for (C) and (D) but their sizes and orientations are different between (C) and (D) because the relevant mirrors are different. PAs belonging to the case B are investigated in DMO and by Wang and Kuo (1988).

We consider only regular approximants associated with the special points of L_s. Several properties of the approximants are listed in table 2. Note, however, that most approximants incur symmetry breaking due to frustrations if the canonical window is used for Q (Niizeki 1991a).

The prototype approximant of the case p4g in (A) is shown in figure 2, while that of p4mm(Γ) in (B) (or cmm(Γ) in (C)) is a square (or rhombic) lattice whose unit cell

Table 1. The index K and its dual J of the four series of the deformed lattices. The first two columns refer to the prototype lattices, and the last two to their kth generations, where $p=u_{k+1}$, $q=u_k$, $r=v_{k+1}$ and $s=v_k$.

	K	J	K_k	J_k
(A)	$[0110/\bar{1}001]$	$(1001/0\bar{1}10)$	$[qppq/\bar{p}\bar{q}qp]$	$(p\bar{q}\bar{q}p/q\bar{p}p\bar{q})$
(B)	$[1000/0010]$	$(0\bar{1}00/000\bar{1})$	$[sq0\bar{q}/0qsq]$	$(q\bar{s}q0/\bar{q}0q\bar{s})$
(C)	$[0100/0010]$	$(0001/1000)$	$[qsq0/0qsq]$	$(q0\bar{q}s/s\bar{q}0q)$
(D)	$[100\bar{1}/1100]$	$(1\bar{1}01/0010)$	$[pq\bar{q}\bar{p}/ppq\bar{q}]$	$(r\bar{p}0p/0\bar{q}s\bar{q})$

Table 2. The space groups and other properties of regular approximants of the four types, A, B, C and D. The Bravais classes of the four are shown in the second column. The second block of columns show the space groups when the phase vectors are located on the special points of L_s (the shadow lattice) as shown in the first row; Γ, M, X and Y are indexed by [00], [hh], [h0] and [0h] with $h=\frac{1}{2}$, respectively. X and Y do not present regular approximants except for the case (D). N_S (or N_R) is the number of the square (or rhombic) tiles in the unit cell, while N $(=N_S+N_R)$ the number of the lattice points.

	Γ	M	X	Y	N_S	N_R	N
(A) p4mm	p4mm	p4g			$2u_{2k+1}$	$2v_{2k+1}$	$2u_{2k+2}$
(B) p4mm	p4mm	p4mm			v_{2k}	$2u_{2k}$	v_{2k+1}
(C) cmm	cmm	cmm			u_{2k}	v_{2k}	u_{2k+1}
(D) cmm	cmm	cmm	cmm	cmm	v_{2k+1}	$2u_{2k+1}$	v_{2k+2}

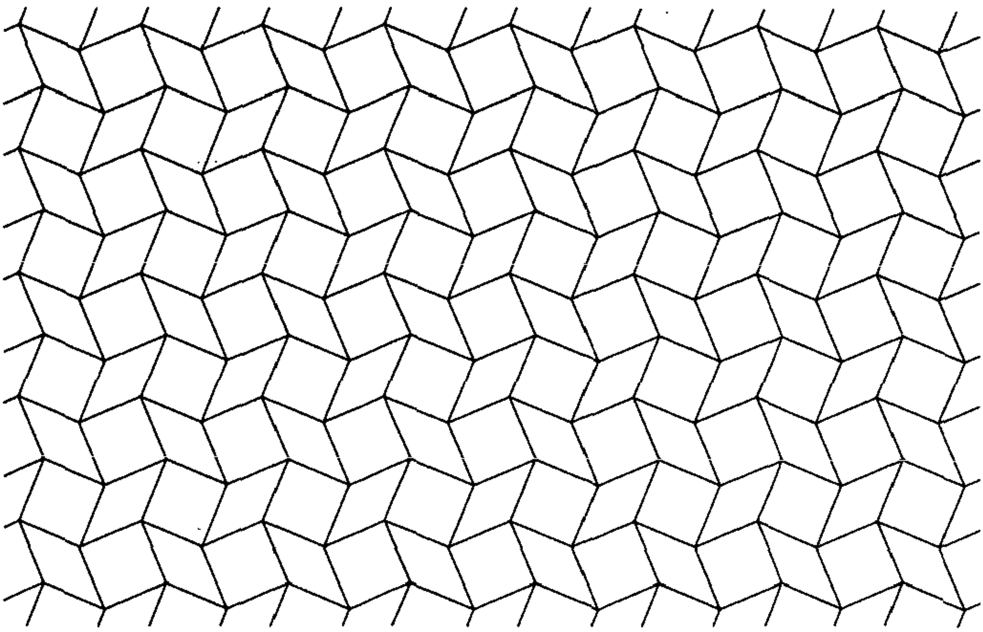

Figure 2. The prototype approximant of the case of p4g in A. The centres of the squares are the special points of the point group 4, while the centres of the rhombi are those of mm. The Bravais lattice is a square lattice formed of the centres of square tiles with a common orientation.

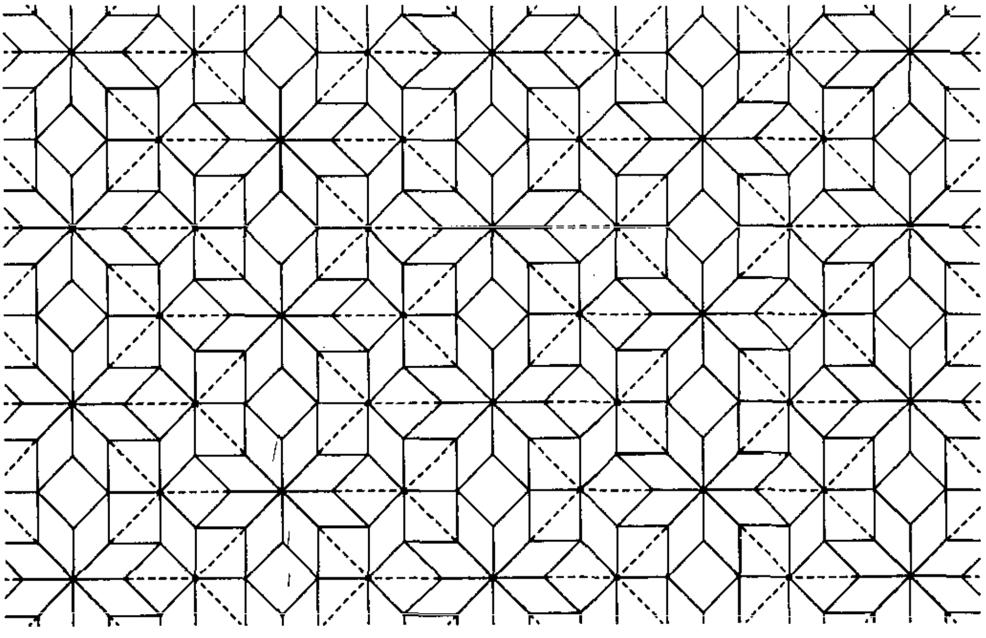

Figure 3. The first generation (solid lines) of the case of pmm(Γ) in D and its inflation (dashed lines). The Bravais lattice is a rhombic lattice formed of the eight-pronged vertices. The inflated lattice represents apart from the scale the prototype of the series. It has one square tile and two rhombic ones per a unit cell.

is identical to the square (or rhombic) tile of the octagonal QL. Figure 2 has been recognized by Mai *et al* (1989) as an approximant to the octagonal QL. The first generation of the case pmm(Γ) in (D) is shown in figure 3 together with its inflation, which is superimposed. The second generations of the cases (A) p4g and (D) cmm(Γ) are shown in figure 4 together with their inflations.

(*a*)

(*b*)

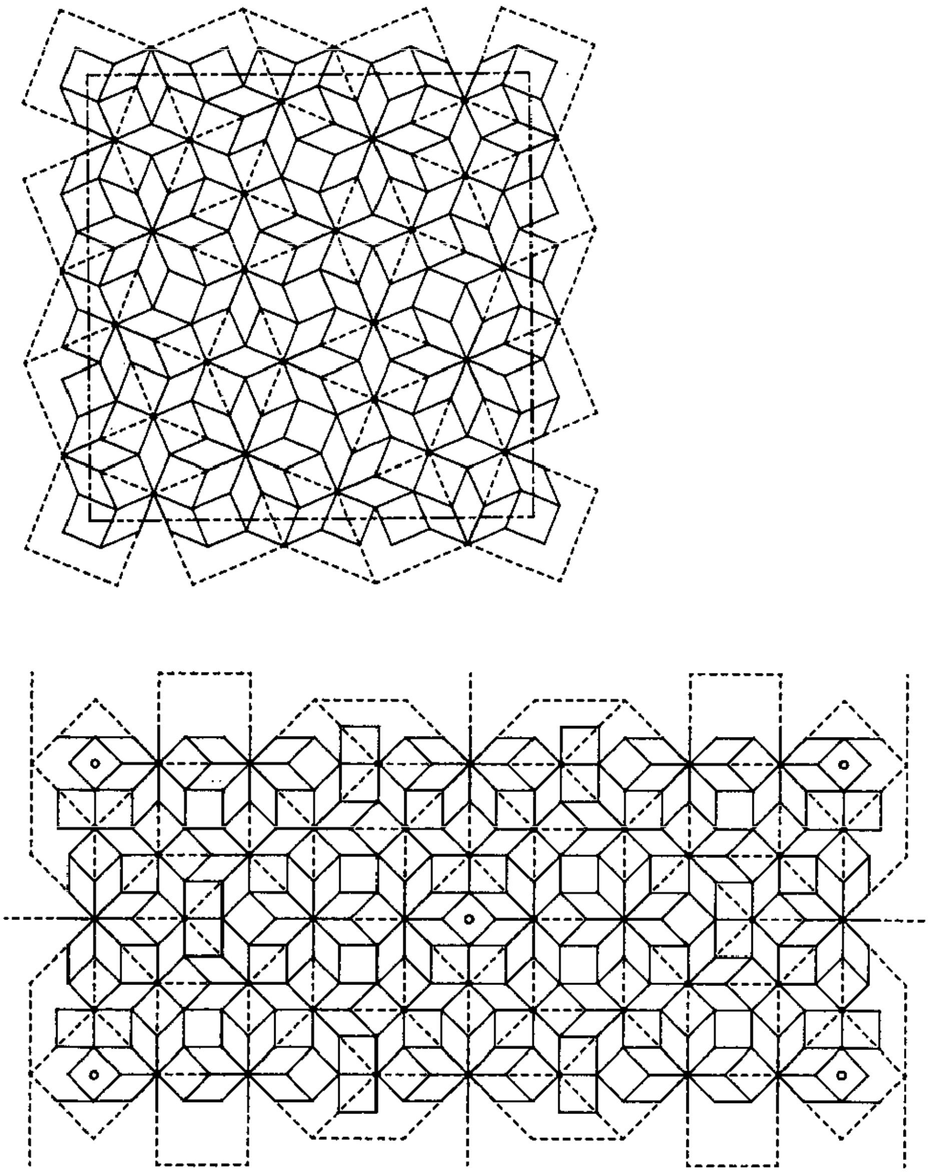

Figure 4. The second generations of the series p4g in A (*a*) and cmm(Γ) in D (*b*) together with their inflations. The chained lines in (*a*) show the unit cell of the Bravais lattice, while the circles in square tiles in (*b*) the lattice points. The special points of (*a*), for example, are derived from those of classes M and C of the mother lattice. The QL formed of the eight-pronged vertices in (*a*) is similar to the prototype shown in figure 2. The inflated QL in (*b*) is similar to the first generation shown in figure 3.

7. Discussions

Let $\mathbf{U}_k$ be the $d \times D$ matrix formed by the internal components of $\tilde{\varepsilon}_i^{(k)}$, the basis vectors of $\tilde{L}_k$. Then, it is written as $\mathbf{U}_k = (\boldsymbol{b}_1^{(k)}\boldsymbol{b}_2^{(k)}\ldots\boldsymbol{b}_d^{(k)})\mathbf{J}_k$, which is rewritten as $\mathbf{U}_k = \mathbf{B}\mathbf{V}_k$, where $\mathbf{B} = (\boldsymbol{b}_1\boldsymbol{b}_2\ldots\boldsymbol{b}_d)$ and $\mathbf{V}_k = (u_k\mathbf{J}' + u_{k-1}\mathbf{J})/\tau^k$ with $\mathbf{J}' = -\mathbf{J}\mathbf{M}^{-1}$. It follows that $\mathbf{V}_k = (\tau_k\mathbf{J}' + \mathbf{J})/(\tau_k\tau + 1)$ with $\tau_k = u_k/u_{k-1}$ because $\tau^k = u_k\tau + u_{k-1}$. τ_k and $\mathbf{U}_k$ tend to τ and $\mathbf{U} = (\boldsymbol{e}_1'\boldsymbol{e}_2'\ldots\boldsymbol{e}_D')$, respectively, as k goes to the infinity, so that we obtain $\mathbf{U} = \mathbf{B}\mathbf{V}$ with $\mathbf{V} = (\tau\mathbf{J}' + \mathbf{J})/(\tau^2+1)$. That is, the row vectors of $\mathbf{V}$ as well as $\mathbf{U}$ are left eigenvectors of $\mathbf{M}$ with respect to its eigenvalue τ'. It is interesting to prove it directly: We begin with the equality

$$\mathbf{B}\mathbf{V} = (\tilde{e}_1'\tilde{e}_2'\ldots\tilde{e}_D')(-\tau\mathbf{M}^{-1} + \mathbf{I})/(\tau^2+1)$$

derived from (9). The right-hand side is equal to $\mathbf{U}$ because we have (5) and the two equalities $(\boldsymbol{e}_1\boldsymbol{e}_2\ldots\boldsymbol{e}_D)\mathbf{M}^{-1} = (\boldsymbol{e}_1\boldsymbol{e}_2\ldots\boldsymbol{e}_D)\tau^{-1}$ and $\mathbf{U}\mathbf{M}^{-1} = -\tau\mathbf{U}$.

The linear independence of $\boldsymbol{e}_i'$ over $\boldsymbol{Z}$ reflects in the expression $\mathbf{U} = \mathbf{B}\mathbf{V}$ in that the incommensurate ratio τ is included in the numerator of $\mathbf{V}$. Since τ_k is a best approximant to τ, we can conclude that $\mathbf{U}_k$ is a best approximant to $\mathbf{U}$. Incidentally, we note that the basis vectors $\boldsymbol{b}_i$ of L_s are determined directly by the equality $\mathbf{U} = \mathbf{B}\mathbf{V}$ if $\mathbf{J}$ is specified.

The case of the dodecagonal QL differs from other cases in that τ', the algebraic conjugate of τ $(=2+\sqrt{3})$, is given by $1/\tau$ but not by $-1/\tau$. The theory developed in this paper is applicable also to this case with a minor modification. The dodecagonal QL has, however, another self-similarity with the scale $\tau_p = (\sqrt{3}+1)/\sqrt{2}$ but accompanied by a rotation through $\pi/12$ (Niizeki 1989a). The theory can be extended so that this self-similarity is included. The result will be published elsewhere.

Acknowledgements

This work is supported by a Grant-in-Aid for Science Research from Ministry of Education, Science and Culture. A part of this work was completed when I was staying at ZiF, University of Bielefeld and I would like to thank Professor L Danzer and Dr F Gahler for their hospitality.

Appendix

We begin with proving the following equality:

$$\tilde{Q}_k(\tilde{\boldsymbol{x}}_k, V) = \tau\tilde{Q}_{k-1}(\tilde{\boldsymbol{x}}_{k-1}, \tau V) \tag{A1}$$

where V is a convex domain satisfying $V = -V$. The condition $\boldsymbol{l} \in \tilde{L}_k$ in the expression of $\tilde{Q}_k(\tilde{\boldsymbol{x}}_k, V)$ (see (13)) is equivalent to $\boldsymbol{l} = \hat{\tau}\boldsymbol{l}'$ with $\boldsymbol{l}' \in \tilde{L}_{k-1}$, so that we obtain

$$\tilde{Q}_k(\tilde{\boldsymbol{x}}_k, V) = \{P(\hat{\tau}\boldsymbol{l} + \tilde{\boldsymbol{x}}_k) \mid \boldsymbol{l} \in \tilde{L}_{k-1}, P'(\hat{\tau}\boldsymbol{l} + \tilde{\boldsymbol{x}}_k) \in V\}. \tag{A2}$$

On the other hand, $P(\hat{\tau}\boldsymbol{l} + \tilde{\boldsymbol{x}}_k) = P(\hat{\tau}(\boldsymbol{l} + \tilde{\boldsymbol{x}}_{k-1})) = \tau P(\boldsymbol{l} + \tilde{\boldsymbol{x}}_{k-1})$ and, similarly, $P'(\hat{\tau}\boldsymbol{l} + \boldsymbol{x}_k) = -\tau^{-1}P'(\boldsymbol{l} + \boldsymbol{x}_{k-1})$ because $\tau' = -1/\tau$. Moreover the condition $-\tau^{-1}P'(\boldsymbol{l} + \boldsymbol{x}_{k-1}) \in V$ is equivalent to $P'(\boldsymbol{l} + \tilde{\boldsymbol{x}}_{k-1}) \in \tau V$. Thus (A1) has been proved.

Using (A1) repeatedly, we arrive at

$$\tilde{Q}_k(\tilde{\boldsymbol{x}}_k, V) = \tau^k\tilde{Q}(\tilde{\boldsymbol{x}}, \tau^k V). \tag{A3}$$

Equation (14) is a special case where $V = \tilde{W}_k$.

The proof of (A1) is based on the two equalities, $\tilde{L}_k = \hat{\tau}\tilde{L}_{k-1}$ and $\tilde{x}_k = \hat{\tau}\tilde{x}_{k-1}$. We can prove (A3) directly in a similar way by using $\tilde{L}_k = (\hat{\tau})^k\tilde{L}$ nd $\tilde{x}_k = (\hat{\tau})^k\tilde{x}$.

Self-similarity of the ideal quasilattice $Q(\boldsymbol{x}, W)$ is based on the equality $Q((\hat{\tau})^k\boldsymbol{x}, W) = \tau^k Q(\boldsymbol{x}, \tau^k W)$, which is proved in a similar way by using the equality $(\hat{\tau})^k L = L$.

References

Duneau M, Mosseri R and Oguey C 1989 *J. Phys. A: Math. Gen.* **22** 4549–64
Elser V and Henley C 1985 *Phys. Rev. Lett.* **55** 2883–6
Ishii Y 1989 *Phys. Rev.* B **39** 11862–71
Janssen T 1988 *Phys. Rep.* **168** 55–113
Katz A and Duneau M 1986 *J. Physique* **47** 181–96
Mai Z H, Xu L, Wang N, Kuo K H, Jin Z C and Cheng G 1989 *Phys. Rev.* B **40** 12183–7
Niizeki K 1989a *J. Phys. A: Math. Gen.* **22** 193–204
—— 1989b *J. Phys. A: Math. Gen.* **22** 4281–93
—— 1990 *J. Phys. A: Math. Gen.* **23** 4569–80
—— 1991a *J. Phys. A: Math. Gen.* **24** 3641–54
—— 1991b *Quasicrystals* ed K H Kuo and T Ninomiya (Singapore: World Scientific) pp 217–23
Sadoc J-F and Mosseri R 1990 *J. Physique* **51** 205–21
Verger-Gaugry J-L 1988 *J. Physique* **49** 1867–74
Wang N and Kuo K H 1988 *Acta Cryst.* A **44** 857–63